REVIEW OF MAT 603 FOR MAT 680

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The topics and problems given below are meant to guide your studying for the comprehensive exam. This is not a contract. There could well be questions over topics not mentioned here. I am making an honest effort to help you as much as I can. I highly suggest making sure you make sure that you have the definitions of any terms mentioned here committed to memory. The same goes for any theorems named here. Once you have these basics under control, consider the simplest examples from each topic. If you can do similar simple exercises, then you should try more challenging problems. If you are struggling with the simple exercises, then you should seek help.

Topics

- Properties of $\mathbb{Q}$ and/vs. $\mathbb{R}$: completeness, least upper bounds, supremums, etc.
- Metric space concepts: verify if something is a metric, open, closed, compact, closures, boundaries, function spaces
- Sequences and series: convergence or divergence, Cauchy criterions, ϵ, N proofs
- Limits of functions: ϵ, δ proofs, limit laws, relation to continuity
- Continuity: ϵ, δ proofs, uniform continuity, fundamental theorem for continuous functions
- Differentiation: proof of differentiability from the definition with or without limit laws, relation to continuity
- Integration: basic concepts, properties, and examples of integrable or non-integrable functions on closed intervals, Riemann-Lebesque theorem, fundamental theorem of calculus
- Power Series: exponential, sine, and cosine functions, radius of convergence, relation to differentiation and integration

Comprehensive Exam Type Problems

The problems below are mostly selected from the midterm and final exams given in my MAT 603 course, taught in Summer 2011. These problems are of varying difficulty and length. Some of the multipart problems are clearly too long for a comprehensive exam question, but selected parts would be reasonable.

(1) (a) Consider the Dedekind cut $x = A \mid B$ where $A = \{r \in \mathbb{Q} : r^3 < -8\}$. Find the least upper bound of A and the greatest lower bound of B. Which of A and B contains its bound?

(b) Describe the closure of $\mathbb{Q}$.

(c) Prove that if A is a nonempty and a proper subset of $\mathbb{R}$ that is open in $\mathbb{R}$, then there is at least one point b that is a limit point of A, but is not an element of A. *Hint 1: Think about A^c. If Hint 1 doesn't help, try Hint 2: How can open sets be described?*

(2) Let M be a metric space with metric d and let $A \subseteq M$.

(a) Define the terms open, closed, and compact with respect to A.

(b) Provide examples to illustrate these terms for metric spaces other than $\mathbb{R}$.

(c) Prove that if A is open, then $M \setminus A$ is closed.

(3) (a) Define what it means for a sequence to converge in a metric space M.

(b) State the Cauchy criterion for sequences in a general metric space M.

(c) Prove that the sequence $a_n = \frac{2n+1}{2n-1}$ converges in $\mathbb{R}$.

(4) (a) Prove that the sequence $b_n = (-2)^n \left(\frac{1}{3^n} + \frac{1}{2^n}\right)$ is bounded, but diverges.

(b) Find a convergent subsequence of (b_n) defined above. Then conjecture its limit and prove your conjecture.

(5) (a) Define the limit supremum of a sequence.

(b) Provide an example of a divergent sequence for which the limit supremum exists, but is never attained by the sequence.

(c) Prove that a sequence is convergent if and only if the limit supremum and limit infimum are equal.

(6) For $n = 0, 1, 2, 3, \ldots$, let $a_n = \left[\frac{4+2(-1)^n}{5}\right]^n$.

(a) Find $\limsup \sqrt[n]{a_n}$, $\liminf \sqrt[n]{a_n}$, $\limsup \left|\frac{a_{n+1}}{a_n}\right|$, and $\liminf \left|\frac{a_{n+1}}{a_n}\right|$.

(b) Does the series $\sum_{n=0}^{\infty} a_n$ converge?

(c) Find the interval of convergence for the power series $\sum_{n=0}^{\infty} a_n x^n$.

(7) (a) Define what it means for a function f to have a limit at a point x.

(b) Use this definition to prove that $f(x) = \frac{2x^2-x-1}{x-1}$ has a limit at $x = 1$.

(c) Define $g(1) = 0$ and $g(x) = \frac{2x^2-x-1}{x-1}$ for $x \neq 1$. Is g continuous at $x = 1$? Explain.

(8) (a) Define what it means for a function $f : D \to \mathbb{R}$ to be continuous at a point $a \in D \subseteq \mathbb{R}$.

(b) Prove that $f(x) = x^2 + x$ is continuous on $\mathbb{R}$.

(9) For each nonzero real number x, let $f(x) = \frac{1}{x}$.

(a) Provide an ϵ, δ proof that f is continuous at each $x > 0$.

(b) For each positive integer n, let $a_n = f(n)$. Provide an ϵ, N proof that (a_n) is convergent.

(10) (a) Prove that $f(x) = \cos(1/x)$ is uniformly continuous on $[a, 1]$ for each $0 < a < 1$. *This proof does not need to be done from the definition.*

(b) Prove that $f(x) = \cos(1/x)$ is not uniformly continuous on $(0, 1)$.

(c) Provide an example of a function $f : (0, 1) \to \mathbb{R}$ that is continuous, but not uniformly continuous on its domain $(0, 1)$.

(d) Provide an example of a uniformly continuous function that is unbounded on $\mathbb{R}$.

(11) (a) Prove, from the definition of uniform continuity, that $f(x) = \cos(x)$ is uniformly continuous on $\mathbb{R}$ if $|\cos x - \cos y| \leq |x - y|$.

(b) Prove, from the definition of uniform continuity, that $f(x) = \frac{x}{x^2+1}$ is uniformly continuous on $\mathbb{R}$.

(c) Provide an example of a continuous function $f : (0, 1) \to \mathbb{R}$ for which ran$f$ is not open in $\mathbb{R}$.

(12) (a) Define what it means for a real valued function to be differentiable.
(b) Define what it means for a real valued function to be integrable.
(c) From these definitions, prove that $f(x) = x^2$ is both differentiable and integrable.
(d) Assume that the product rule is true. Then use induction to prove that $f_n(x) = x^n$ is differentiable on $\mathbb{R}$ for any $n \in \mathbb{N}$.

(13) (a) State the fundamental theorem of calculus. And then explain it.
(b) Find nontrivial upper and lower bounds for $F(x) = \int_e^{x^2} (\ln t)^5$ if $x > e$. You may use the interval itself as the partition to make the calculations easy.
(c) Find $F'(x)$.

(14) For each real number x, let $f(x)$ be the least integer less than or equal to x.
(a) Find $\int_0^5 f(x)dx$ and $\int_0^{5.1} f(x)dx$. *Do this in the easiest possible way.*
(b) Use the definition of the derivative to find $F'(5.1)$. *You may use limit laws.*
(c) Show that $F'(5)$ does not exist. *You may use limit laws.*
(d) Explain why F is continuous at $x = 5$. *You may use limit laws.*
(e) Prove that if $f(x)$ is integrable over the interval $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is differentiable

Additional True/False Questions to Help You Study

(1) In $\mathbb{R}$, every open set is the countable union of open intervals.
(2) The interval $[3, 5)$ is both open and closed in $\mathbb{R}$.
(3) If $f : M \to N$ is a continuous function between metric spaces and M is compact, then N is compact.
(4) The rational number are the only dense subset of $\mathbb{R}$.
(5) A function is differentiable if and only if it is integrable.
(6) If $x = A|B$ and $y = C|D$ are two Dedekind cuts, then $x + y$ is the Dedikind cut $(A \cup B)|(C \cup D)$.