MAT 680: MAT 501 - Fundamental Concepts of Mathematics

Some of the highlights of MAT 501:

- **Propositional calculus** - truth tables; negation; conjunction; disjunction; implication; biconditional; converse; contrapositive
- **Proof techniques** - direct proofs; cases; contradiction; contrapositive; induction
- **Set Theory** - basic set theory; subsets; unions; intersections; de Morgan's laws; collections of sets; power sets
- **Relations** - definition of relation; reflexive, symmetric, antisymmetric, transitive relations; equivalence relations; partial orderings; inverse relations
- **Functions** - definition of function; image; preimage; injective, surjective, bijective functions; inverse functions
- **Number systems** - natural numbers and well-ordering principle; integers; rational numbers; real and complex numbers

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(1) Using a truth table, show that the statement $P \Rightarrow (Q \wedge R)$ is logically equivalent to $(P \Rightarrow Q) \wedge (P \Rightarrow R)$.

(2) (a) Prove that if a^2 is even, then a is even.
(b) Prove: If x is rational and nonzero and y is irrational, then xy is irrational.

(3) Let f_n denote the nth Fibonacci number (so $f_0 = 0$, $f_1 = 1$, and $f_k = f_{k-1} + f_{k-2}$ for all $k \geq 2$). Prove that $f_n < 2^n$ for all $n \in \mathbb{N}$.

(4) (a) Prove $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.
(b) Does an anologous result hold for unions of sets? Prove or disprove.

(5) Let U be a nonempty set and define a relation R on $\mathcal{P}(U)$ via

$$A\,R\,B \Leftrightarrow A - B \text{ is finite.}$$

Determine which of the following properties R satisfies: reflexive, symmetric, antisymmetric, transitive.

(6) Let $f : X \to Y$ be a function and let $Z \subseteq X$.
(a) Prove that $f(X) - f(Z) \subseteq f(X - Z)$.
(b) Show by counterexample that equality need not hold in (a).

(7) Let $f : \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ x^2 + 1 & \text{if } x < 0 \end{cases}$$

Is f injective, surjective, bijective? Prove your assertions.

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(8) Let $f: X \to Y$ be a function and A, B subsets of Y. What is the relationship between $f^{-1}(A) \cap f^{-1}(B)$ and $f^{-1}(A \cap B)$?

Write a statement using these sets and one of the symbols $\subseteq$, $\supseteq$, or $=$, and prove your assertion. If you claim that equality does not hold, provide a counterexample.

(9) Let $I \neq \emptyset$, $U \neq \emptyset$ and $\{B_i | i \in I\}$ a collection of subsets of U. Prove that

$$\left(\bigcap_{i \in I} B_i\right)^C = \bigcup_{i \in I} (B_i^C)$$

where X^C denotes the complement of X in U.

(10) Negate the statement $\forall \varepsilon > 0 \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Disclaimer: None of these problems should be construed as being comprehensive exam questions. They are merely meant to illustrate some of the main concepts of MAT 501.