REVIEW: MAT 603 REAL ANALYSIS

The following is an overall guideline to help you study for MAT 603 for the Graduate Comprehensive Exam. You will be tested on your knowledge of definitions, terminology, proof writing and problem solving techniques, understanding of the concept and, proofs of basic as well as standard results from the topics mentioned below.

Please note that the topics and problems below are meant to illustrate some of the main concepts of MAT 603. They should not necessarily be considered as examples of what is going to be on the comprehensive exam. It is your responsibility to study topics listed below in more detail, and practice more related problems.

- **Real Number System:** Ordered Field Axioms, Completeness Axiom (or the Least Upper Bound Property), Properties of $\mathbb{R}$ and $\mathbb{Q}$ which follow from these axioms, such as Archimedean Property, $\mathbb{Q}$ dense in $\mathbb{R}$, etc., Supremum and Infimum of a set, Dedekind cuts and their properties.

- **Sequences in $\mathbb{R}$:** $\epsilon - N$ proof of convergence and divergence of sequences in $\mathbb{R}$, basic properties, Cauchy sequences.

- **Limits and Continuity in $\mathbb{R}$:** $\epsilon - \delta$ proofs of limits, continuity, and uniform continuity, basic properties of limits and continuous functions, sequential criterion for limits and continuity.

Students must have a thorough knowledge of any two of the following topics:

- **Metric Spaces:** Proving a given function is a metric, definition, examples and basic results pertaining to open sets, closed sets, open balls, limit points of a set, closure of a set, compact sets.

- **Differentiation:** Definition and basic properties of derivatives, proof of differentiability of a function using the definition, relation to continuity.

- **Riemann Integration:** Definition and basic properties, examples, Fundamental Theorem of Calculus.

Some Practice Problems

1. Prove that the following are metrics on $\mathbb{R}^2$:

   (i) $d((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

   (ii) $d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

2. If $d : X \to [0, \infty)$ is a metric on X, then prove that so is D, where $D(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$.

3. Consider following sets in $\mathbb{R}$:

   (i) $\mathbb{N}$ (ii) $\mathbb{Q}$ (iii) $(0, 1)$ (iii) $[-1, 2]$ (iv) $(2, 3]$

   (v) $\{3, -1, 0\}$ (vi) $[2, 4] \cup \{1\}$ (vii) $\displaystyle\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right)$ (viii) $\displaystyle\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right)$

   (a) Classify these sets as either only closed, only open, both closed and open, or neither, in $\mathbb{R}$. Justify your answer.

   (b) Find the set of all limit points of these sets, and find their closure.

4. Prove that the following limits exist, using $\epsilon - N$ definition of convergence of a sequence in $\mathbb{R}$.

   (i) $\displaystyle\lim_{n\to\infty} \frac{5x_n - 3}{2x_n + 4} = \frac{5}{2}$

   (ii) $\displaystyle\lim_{n\to\infty} \frac{4x_n^2 + 7}{x_n^2 + 1} = 4$

   (iii) $\displaystyle\lim_{n\to\infty} \frac{1}{3^n} = 0$.

5. Prove that the sequence (x_n) where $x_n = \cos \dfrac{n\pi}{3}$, diverges.

6. Prove that the following functions are continuous on $\mathbb{R}$, using $\epsilon - \delta$ definition of continuity.

   (i) $f(x) = 5x - 9$

   (ii) $f(x) = x^2 + 2x$

   (iii) $f(x) = \sqrt{x}$

7. Let $f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

   Show that f is continuous at $x = 0$, and is discontinuous at all $x \neq 0$.

   [Hint: Use sequential criterion.]

8. Prove that $f(x) = 3x - 6$ is uniformly continuous on $\mathbb{R}$.

9. Let $f(x) = x^2$, then prove that f is uniformly continuous on $[0, a]$ but not uniformly continuous on $[0, \infty)$.

10. Let α, β and γ be cuts. Prove that

    (i) $\alpha + \beta := \{q \in \mathbb{Q} : \; q = r + s \text{ for some } r \in \alpha, s \in \beta\}$ is a cut.

    (ii) Exactly one of $\alpha > \beta$, $\alpha = \beta$, $\beta > \alpha$ is true.

    (iii) If $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \beta$

    (iv) $\alpha + 0^* = \alpha$, where 0^* is the rational cut corresponding to 0.

11. Using definition of a derivative, prove the following functions are differentiable, and find their derivative:

    (i) $f(x) = 2x - 1$ (ii) $f(x) = \dfrac{1}{x}$ (iii) $f(x) = \dfrac{1}{\sqrt{x}}$.

12. Prove that $f(x) = x^{1/3}$ is differentiable at all $x \neq 0$, with derivative $\dfrac{1}{3}x^{-2/3}$, and it is not differentiable at $x = 0$.

13. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as: $f(x) = \begin{cases} x^2 & \text{if x is a rational} \\ 0 & \text{if x is an irrational} \end{cases}$

    Then prove that f is differentiable at $x = 0$.

14. Using the Fundamental Theorem of Calculus, find a formula for the derivatives of each function:

    (i) $F(x) = \int_0^{\sin x} \cos t^2 dt$ (ii) $F(x) = \int_x^{x^2} \sqrt{1 + t^2} dt$

15. Let $f(t) = \begin{cases} t & 0 \leq t \leq 2 \\ 3 & 2 < t \leq 4 \end{cases}$

    (a) Find an explicit expression for $F(x) = \int_0^x f(t)dt$ as a function of x.

    (b) Sketch F and determine where F is differentiable.

    (c) Find a formula for $F'(x)$ wherever F is differentiable.