MAT 680: MAT 604 - Algebraic Structures II

Main topics of MAT 604:

- **Rings and fields** - definition and basic properties; integral domains; field of quotients
- **Polynomial rings** - reducible and irreducible polynomials over different rings and fields; factoring polynomials into irreducibles
- **Ideals** - definition and basic properties; prime ideals; maximal ideals; principal ideals
- **Homomorphisms and factor rings** - definition of a ring homomorphism; cosets in a ring and how to add/multiply them
- **Field extensions** - extension fields; degree of a field extension and an element; algebraic and transcendental extensions

Disclaimer: None of the problems on this handout should be construed as being comprehensive exam questions, nor do they necessarily cover the entire spectrum of possible questions. They are merely meant to illustrate some of the main concepts of MAT 604.

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(1) Let $R = \{0, 2, 4, 6, 8, 10, 12\}$ with operations addition mod 14 and multiplication mod 14.

(a) Prove that R is a commutative ring.

(b) Does R have a multiplicative identity? Is R an integral domain? A field? Justify your answers.

(2) Let $H = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} : a, b \in \mathbb{Q} \right\}$. Show that H is a ring under the usual operations of matrix addition and matrix multiplication. You may assume that $M_2(\mathbb{R})$, the set of 2×2 matrices with entries in $\mathbb{R}$ is a ring under these operations.

(3) Let $R = \mathbb{Q}[x]$, the ring of polynomials with coefficients in $\mathbb{Q}$. Let

$$I = \{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x \mid n \in \mathbb{N}, a_i \in \mathbb{Q}\},$$

i.e., the set of polynomials in R with zero constant term.

(a) Prove that I is an ideal of R.

(b) Is I a prime ideal and/or a maximal ideal of R? Justify your answers.

(4) Construct a field with 9 elements and give its addition and multiplication tables.

(5) (a) Let i be the imaginary unit (so $i^2 = -1$). Use a field degree argument (with an appropriate diagram) to prove that $i \notin \mathbb{Q}(\sqrt{-2})$.

(b) Find the degree of $\sqrt[3]{2}$ over $\mathbb{Q}(\sqrt{2})$. Justify your answer.

(6) Factor each of these polynomials over the given fields into irreducible polynomials.

If you end up with any irreducible polynomials other than linear ones (degree 1), explain why they are irreducible.

(a) $x^2 + x + 1$ over $\mathbb{Z}_2$

(b) $x^4 - 2x^2 + 1$ over $\mathbb{Z}_3$

(c) $x^4 - 4$ over $\mathbb{Q}$

(7) (a) Prove that if $\varphi : F_1 \to F_2$ is a nontrivial homomorphism of fields, then $\varphi(1_{F_1}) = 1_{F_2}$.

(b) Given an example of rings R_1, R_2 with 1 and a nontrivial ring homomorphism $\varphi : R_1 \to R_2$ such that $\varphi(1_{R_1}) \neq 1_{R_2}$.

(8) (a) Determine the irreducible polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.

(b) Find the degree $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}]$

(c) Give a basis for $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ over $\mathbb{Q}$

(9) (a) Let S_1 and S_2 be subrings of a ring R. Prove that $S_1 \cap S_2$ is a subring of R.

(b) For any ring R and any fixed $a \in R$, define R_a to be the subring of R that is the intersection of all subrings of R containing a. Find a simple description for R_5 as a subring of $R = \mathbb{Z}$ and justify your findings.

(10) Let R be an arbitrary ring and $a, b, c \in R$. Prove or disprove:

(a) $a^2 = a$ implies $a = 0$ or $a = 1$.

(b) $ab = 0$ implies $a = 0$ or $b = 0$.

(c) $ab = ac$ and $a \neq 0$ implies $b = c$.

(11) Let I and J be ideals of a ring R. The product IJ of I and J is defined by

$$IJ = \left\{ \sum_{i=1}^{n} x_i y_i : x_i \in I, y_i \in J, n \in \mathbb{N} \right\}.$$

(a) Prove that IJ is an ideal of R.

(b) Prove that $IJ \subseteq (I \cap J)$.